

CERTAIN IMPRIMITIVE REFLECTION GROUPS AND THEIR GENERIC VERSIONS

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Dedicated to Professor Cao Xi-hua on his 80th birthday

ABSTRACT. The present paper is concerned with the connection between the imprimitive reflection groups $G(m, m, n)$, $m \in \mathbb{N}$, and the affine Weyl group \tilde{A}_{n-1} . We show that \tilde{A}_{n-1} is a generic version of the groups $G(m, m, n)$, $m \in \mathbb{N}$. We introduce some new presentations of these groups which are shown to have some group-theoretic advantages. Then we define the Hecke algebras of these groups and of their braid versions, each in two ways according to two presentations. Finally we give a new description for the affine root system $\bar{\Phi}$ of \tilde{A}_{n-1} such that the action of \tilde{A}_{n-1} on $\bar{\Phi}$ is compatible with that of $G(m, m, n)$ on its root system in some sense.

Reflection groups have well-known classical presentations (see [2], [3], [5]). This gives a neat way to describe the reflection groups. But there might be some limitations. Here are two examples. One is for the imprimitive reflection groups $G(m, m, n)$, $m, n \in \mathbb{N}$ (the set of positive integers) defined in 1.2. With respect to the presentation of $G(m, m, n)$ given in [2] (called the standard presentation of $G(m, m, n)$, see Fig.1), there is no known formula for its length function (see [1]). The other is for the affine Weyl groups \tilde{A}_{n-1} , $n > 1$, which form an infinite family of reflection groups (here and later we denote by \tilde{A}_{n-1} the affine Weyl group of type \tilde{A}_{n-1} , by abuse of notation). With respect to the presentation of \tilde{A}_{n-1} given in [5, 4.7] (called the standard presentation of \tilde{A}_{n-1} , see Fig.3), there is no natural way to regard \tilde{A}_h as a subgroup of \tilde{A}_k for $h < k$, comparing with the case of symmetric groups. Then it is natural to ask if there exist some other presentations of $G(m, m, n)$ and \tilde{A}_{n-1} , again by generators and relations, such that the above obstructions could be overcome. In the present paper, we are concerned with the connection between the groups $G(m, m, n)$, $m \in \mathbb{N}$, and their generic version $G(\infty, \infty, n)$ (which is isomorphic to \tilde{A}_{n-1}). We give a new presentation for each of the groups $G(m, m, n)$ and $G(\infty, \infty, n)$. Then we use them to study these groups, the associated braid groups, Hecke algebras and root systems. As a consequence, an affirmative answer is given to the above question.

A generic version $G(\infty, \infty, n)$ for the groups $G(m, m, n)$, $m \in \mathbb{N}$, is a certain monomial matrix group over the set $\{x^k \mid k \in \mathbb{Z}\}$ (x an indeterminate) such that

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any $G(m, m, n)$ is its homomorphic image under the specialization $x = e^{2\pi i/m}$ (see 1.5). $G(\infty, \infty, n)$ is isomorphic to a subgroup in the inverse limit of the inverse system formed by the groups $G(m, m, n)$, $m \in \mathbb{N}$ (see 1.6). $G(\infty, \infty, n)$ can be regarded as the matrix form of \tilde{A}_{n-1} (see Theorem 2.3) and is “the Weyl group” of some algebraic group over the ring $\mathbf{k}[x, x^{-1}]$, where \mathbf{k} is an algebraically closed field of positive characteristic (see [6] and 2.6). This form of \tilde{A}_{n-1} is useful in the study of its conjugacy problem (see [9, Sect. 3]).

For each of the groups $G(\infty, \infty, n)$ and $G(m, m, n)$, we introduce a new presentation by generators and relations (see Proposition 3.3, and 3.4 (2)). The new presentation of $G(\infty, \infty, n)$ (resp. $G(m, m, n)$) is compatible with the standard one of $G(m, m, n)$ (resp. $G(\infty, \infty, n)$) under the specialization map $\phi_m^n : G(\infty, \infty, n) \rightarrow G(m, m, n)$. The new presentation of $G(\infty, \infty, n)$ has the advantage that a lower rank member in the family is naturally a subgroup of any higher rank member. Also, the new presentation of $G(m, m, n)$ has the advantage that the length function can be formulated explicitly (see 3.5).

We define the Hecke algebras associated to the groups $G(\infty, \infty, n)$, $G(m, m, n)$ and the associated braid groups, each in two ways according to two presentations (standard and new), which are shown to be isomorphic (see 4.1, 4.2 and Proposition 4.3). Moreover, the Hecke algebra of $G(m, m, n)$ (resp. $B(m, m, n)$) can be naturally regarded as a quotient of the Hecke algebra $\mathcal{H}(G(\infty, \infty, n))$ (resp. $\mathcal{H}(B(\infty, \infty, n))$) (see 4.4). Note that our definition for the Hecke algebra of $G(m, m, n)$ is slightly different from that of [2], the former involves one parameter and the latter involves two.

We describe the root system $\bar{\Phi}$ of $G(\infty, \infty, n)$. $\bar{\Phi}$ is essentially the affine root system of type \tilde{A}_{n-1} on which the group $G(\infty, \infty, n)$ acts naturally. It has the additional property that the action of $G(\infty, \infty, n)$ on $\bar{\Phi}$ is compatible with that of $G(m, m, n)$ on its root system $\Phi(m)$ via a pre-root system Φ and the specialization $x = e^{2\pi i/m}$ (see 5.2 and 5.5).

The connection between the groups $G(\infty, \infty, n)$ and $G(m, m, n)$ makes it possible to generalize some results which were originally shown only for one of the groups $G(\infty, \infty, n)$ and $G(m, m, n)$, but now could be shared by both. For example, such results can be found easily in the representation theory, since any representation of $G(m, m, n)$ could be lifted to $G(\infty, \infty, n)$.

An analogous discussion can proceed on the imprimitive groups $G(m, 1, n)$, $m \geq 1$, and their generic version $G(\infty, 1, n)$. We shall deal with it in a forthcoming paper.

The content of the paper is organized as follows. In Section 1, we introduce the generic version $G(\infty, 1, n)$ (resp. $G(\infty, \infty, n)$) for the imprimitive reflection groups $G(m, 1, n)$ (resp. $G(m, m, n)$). Then we concentrate ourselves only on $G(\infty, \infty, n)$ and $G(m, m, n)$ in the subsequent sections. We show the isomorphism $G(\infty, \infty, n) \cong \tilde{A}_{n-1}$ in Section 2. For each of $G(\infty, \infty, n)$ and $G(m, m, n)$, we introduce a new presentation by generators and relations in Section 3, and two associated Hecke algebras in Section 4; the latter are shown to be isomorphic. Finally, we give a new description for the affine root system of $G(\infty, \infty, n)$ in Section 5.

1. THE GENERIC VERSIONS OF CERTAIN IMPRIMITIVE REFLECTION GROUPS

Fix $n \in \mathbb{N}$. In the present section, we introduce the generic versions for two families of imprimitive reflection groups acting on \mathbb{C}^n .

1.1. Let V be an n -dimensional complex vector space with a unitary inner product (\mid) . A *reflection* in V is by definition a unitary transformation σ of V of finite order with $\dim(1 - \sigma)V = 1$. A group G generated by reflections in V is called a *reflection group*. A reflection group G is called *Coxeter*, if there exists a G -invariant \mathbb{R} -subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \rightarrow V$ is bijective. A reflection group G in V is *imprimitive*, if there exists a decomposition $V = V_1 \oplus \cdots \oplus V_r$ of nontrivial proper subspaces V_i , $1 \leq i \leq r$, of V such that G permutes $\{V_i \mid 1 \leq i \leq r\}$ (see [3]).

1.2. Let S_n be the symmetric group on n letters $1, 2, \dots, n$. For $\sigma \in S_n$ with $\sigma(i) = r_i$, $1 \leq i \leq n$, we denote by

$$[a_1, \dots, a_n \mid r_1, \dots, r_n] \quad \text{or} \quad [a_1, \dots, a_n \mid \sigma]$$

the $n \times n$ monomial matrix with non-zero entries a_i in the (i, r_i) -positions. For $p \mid m$ (read “ p divides m ”) in \mathbb{N} , we set

$$G(m, p, n) = \left\{ [a_1, \dots, a_n \mid \sigma] \in \text{GL}(\mathbb{C}, n) \mid a_i^m = 1; \left(\prod_j a_j \right)^{m/p} = 1; \sigma \in S_n \right\},$$

where $\text{GL}(\mathbb{C}, n)$ is the group of all $n \times n$ invertible matrices over the complex number field \mathbb{C} . $G(m, p, n)$ is an imprimitive reflection group acting on \mathbb{C}^n , which is Coxeter only when either $m \leq 2$ or $(m, p, n) = (m, m, 2)$. We have $G(m, p, n) = G(1, 1, n) \ltimes A(m, p, n)$, where $A(m, p, n)$ consists of all the diagonal matrices of $G(m, p, n)$, and $G(1, 1, n) \cong S_n$.

1.3. We have $G(m, m, n) \subseteq G(m, 1, n)$ in general. For $q \mid m$ in \mathbb{N} , define a map

$$\begin{aligned} \phi_{m,q}^n : G(m, 1, n) &\rightarrow G(q, 1, n), \\ (a_{ij}) &\mapsto (a_{ij}^{m/q}). \end{aligned}$$

Then we have $\phi_{l,q}^n = \phi_{m,q}^n \phi_{l,m}^n$ if $q \mid m$ and $m \mid l$.

1.4. Let x be an indeterminate. Define

$$(1.4.1) \quad G(\infty, 1, n) = \left\{ [x^{k_1}, \dots, x^{k_n} \mid \sigma] \mid k_i \in \mathbb{Z}; \sigma \in S_n \right\},$$

$$(1.4.2) \quad G(\infty, \infty, n) = \left\{ [x^{k_1}, \dots, x^{k_n} \mid \sigma] \in G(\infty, 1, n) \mid \sum_j k_j = 0 \right\}.$$

Then $G(\infty, 1, n)$ and $G(\infty, \infty, n)$ are two matrix groups. In particular, $G(\infty, \infty, n)$ is a normal subgroup of $G(\infty, 1, n)$ with the quotient $G(\infty, 1, n)/G(\infty, \infty, n) \cong (\mathbb{Z}, +)$.

1.5. $x = e^{2\pi i/m}$ specialization gives a surjection

$$\phi_m^n : G(\infty, 1, n) \twoheadrightarrow G(m, 1, n).$$

By restriction, we also get surjections

$$\begin{aligned} \phi_m^n : G(\infty, \infty, n) &\twoheadrightarrow G(m, m, n), \\ \phi_{m,q}^n : G(m, m, n) &\twoheadrightarrow G(q, q, n) \quad \text{for } q \mid m \text{ in } \mathbb{N}. \end{aligned}$$

We have $\phi_m^n = \phi_{l,m}^n \phi_l^n$ on both $G(\infty, 1, n)$ and $G(\infty, \infty, n)$ for $m \mid l$. So $G(\infty, 1, n)$ (resp. $G(\infty, \infty, n)$) can be regarded as a generic version of the $G(m, 1, n)$'s (resp. $G(m, m, n)$'s).

1.6. We have two inverse systems:

$$(1.6.1) \quad (\{G(m, 1, n)\}_{m \in \mathbb{N}}, \{\phi_{l,m}^n\}_{m|l \text{ in } \mathbb{N}}),$$

$$(1.6.2) \quad (\{G(m, m, n)\}_{m \in \mathbb{N}}, \{\phi_{l,m}^n\}_{m|l \text{ in } \mathbb{N}}).$$

Let $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} (\hat{\mathbb{Z}}_p, +)$ be the direct product of the additive group $(\hat{\mathbb{Z}}_p, +)$ of the completion of the p -adic integer ring \mathbb{Z}_p , whose composition is written multiplicatively. Let U_m be the multiplicative group generated by the m th root $\zeta_m = e^{2\pi i/m}$ of unity for $m \in \mathbb{N}$. Define a map $\psi_m : \hat{\mathbb{Z}} \rightarrow U_m$ as follows. Let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the factorization into the product of distinct prime powers $p_i^{k_i}$. Then for any $a = \prod_p a_p \in \hat{\mathbb{Z}}$ with $a_p \in \hat{\mathbb{Z}}_p$, define $\psi_m(a) = \zeta_m^k$, where the integer k is uniquely determined by the conditions $0 \leq k < m$ and $k \equiv a_{p_i} \pmod{p_i^{k_i}}$ for $1 \leq i \leq r$ according to the Chinese remainder theorem. Let

$$(1.6.3) \quad G(\hat{\mathbb{Z}}, 1, n) = \left\{ [a_1, \dots, a_n | \sigma] \mid a_i \in \hat{\mathbb{Z}}; \sigma \in S_n \right\},$$

$$(1.6.4) \quad G(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}, n) = \left\{ [a_1, \dots, a_n | \sigma] \mid a_i \in \hat{\mathbb{Z}}; \prod_i a_i = 1; \sigma \in S_n \right\}.$$

Define a map $\psi_m^n : G(\hat{\mathbb{Z}}, 1, n) \rightarrow G(m, 1, n)$ (resp. $G(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}, n) \rightarrow G(m, m, n)$) by sending (a_{ij}) to $(\psi_m(a_{ij}))$. One can easily check the equation $\psi_m^n = \phi_{l,m}^n \psi_l^n$ for any $m|l$ in \mathbb{N} . Then the inverse limit of the system (1.6.1) is $(G(\hat{\mathbb{Z}}, 1, n), \{\psi_m^n\}_{m \in \mathbb{N}})$.

Also, the inverse limit of the system (1.6.2) is $(G(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}, n), \{\psi_m^n\}_{m \in \mathbb{N}})$. The group $G(\infty, 1, n)$ (resp. $G(\infty, \infty, n)$) can be embedded into $G(\hat{\mathbb{Z}}, 1, n)$ (resp. $G(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}, n)$) by the specialization $x = \mathbf{e}$ (the element of $\hat{\mathbb{Z}}$ whose component in $\hat{\mathbb{Z}}_p$ is 1 for any prime p).

In the subsequent sections, we shall concentrate only on the groups $G(m, m, n)$, $m \in \mathbb{N}$, and their generic version $G(\infty, \infty, n)$.

2. THE ISOMORPHISM BETWEEN $G(\infty, \infty, n)$ AND \tilde{A}_{n-1}

In this section, we first show the group isomorphism $G(\infty, \infty, n) \cong \tilde{A}_{n-1}$. Then we deduce a formula for the length function of $G(\infty, \infty, n)$ as a Coxeter group, which will be used in 3.5 to formulate a length function of the reflection group $G(m, m, n)$. Finally we show that $G(\infty, \infty, n)$ can be regarded naturally as a “Weyl group” for a certain algebraic group.

2.1. The affine Weyl group \tilde{A}_{n-1} , $n > 1$, has the permutation form (see [8, Sect. 4.1])

$$(2.1.1) \quad \left\{ w : \mathbb{Z} \rightarrow \mathbb{Z} \mid (i+n)w = (i)w + n, \forall i \in \mathbb{Z}; \sum_{i=1}^n (i)w = \sum_{i=1}^n i \right\}.$$

We have $\tilde{A}_{n-1} \cong W_0 \ltimes N$ with $N = \{w \in \tilde{A}_{n-1} \mid (i)w \equiv i \pmod{n}, \forall i \in \mathbb{Z}\}$ and $W_0 = \{w \in \tilde{A}_{n-1} \mid ([1, n])w = [1, n]\} \cong S_n$, where $[a, b] := \{a, a+1, \dots, b\}$ for $a \leq b$ in \mathbb{Z} .

The group \tilde{A}_{n-1} is generated by $S = \{t_0, t_1, \dots, t_{n-1}\}$, where

$$(i)t_k = \begin{cases} i, & \text{if } i \not\equiv k, k+1 \pmod{n}, \\ i+1, & \text{if } i \equiv k \pmod{n}, \\ i-1, & \text{if } i \equiv k+1 \pmod{n}. \end{cases}$$

2.2. To each $w \in \tilde{A}_{n-1}$, we associate an element

$$\eta(w) = [x^{g_1}, x^{g_2}, \dots, x^{g_n} | r_1, r_2, \dots, r_n]$$

of $G(\infty, 1, n)$ if $(i)w = g_i n + r_i$ with $g_i, r_i \in \mathbb{Z}$ and $r_i \in [1, n]$ for $i \in [1, n]$. By (1.4.2) and (2.1.1), we have $\eta(w) \in G(\infty, \infty, n)$. Thus η is a map from \tilde{A}_{n-1} to $G(\infty, \infty, n)$.

Now we are given an element $X = [x^{h_1}, x^{h_2}, \dots, x^{h_n} | u_1, u_2, \dots, u_n]$ of $G(\infty, \infty, n)$. We associate to X a permutation $\kappa(X)$ on \mathbb{Z} as follows. Write $m = qn + r$ with $q \in \mathbb{Z}$ and $r \in [1, n]$ for any $m \in \mathbb{Z}$. We set

$$(m)\kappa(X) = (q + h_r)n + u_r.$$

It is easily seen that $\kappa(X)$ is an element of \tilde{A}_{n-1} . Moreover, we see that the maps η and κ are inverse to each other.

Theorem 2.3. η is a group isomorphism from \tilde{A}_{n-1} to $G(\infty, \infty, n)$.

Proof. We have shown that η is bijective. It remains to show that η preserves the group multiplication. That is, to show $\eta(w)\eta(y) = \eta(wy)$ for any $w, y \in \tilde{A}_{n-1}$. For $i, j \in [1, n]$, write $(i)w = g_i n + r_i$ and $(j)y = h_j n + u_j$ with $g_i, h_j \in \mathbb{Z}$ and $r_i, u_j \in [1, n]$. Then $(i)(wy) = (g_i + h_{r_i})n + u_{r_i}$. This implies that the (i, m) -entry of the matrix $\eta(wy)$ is given by

$$\eta(wy)_{im} = \begin{cases} 0, & \text{if } m \neq u_{r_i}, \\ x^{g_i + h_{r_i}}, & \text{if } m = u_{r_i}. \end{cases}$$

On the other hand, we have

$$\begin{aligned} \eta(w)_{im} &= \begin{cases} 0, & \text{if } m \neq r_i, \\ x^{g_i}, & \text{if } m = r_i. \end{cases} \\ \eta(y)_{jl} &= \begin{cases} 0, & \text{if } l \neq u_j, \\ x^{h_j}, & \text{if } l = u_j. \end{cases} \end{aligned}$$

This implies

$$(\eta(w)\eta(y))_{ip} = \begin{cases} 0, & \text{if } p \neq u_{r_i}, \\ x^{g_i + h_{r_i}}, & \text{if } p = u_{r_i}, \end{cases}$$

So we get $\eta(w)\eta(y) = \eta(wy)$, as required. \square

By Theorem 2.3, we can identify \tilde{A}_{n-1} with $G(\infty, \infty, n)$. Call $\eta(w)$ the matrix form of $w \in \tilde{A}_{n-1}$.

2.4. We have

$$\eta(t_k) = [1, \dots, 1|(k, k+1)] \quad \text{for } k \in [1, n-1],$$

and

$$\eta(t_0) = [x^{-1}, 1, \dots, 1, x|(1, n)],$$

where the notation (i, j) stands for the transposition of i and j . Moreover, the image $\eta(N)$ consists of all the diagonal matrices in $G(\infty, \infty, n)$, and $\eta(W_0) = G(1, 1, n)$.

The length function ℓ of the Coxeter system (\tilde{A}_{n-1}, S) can be formulated in terms of permutation forms and of matrix forms respectively as follows.

Proposition 2.5. (1) $\ell(w) = \sum_{1 \leq i < j \leq n} \left| \left[\frac{(j)w - (i)w}{n} \right] \right|$ for $w \in \tilde{A}_{n-1}$.

(2) For $X = [x^{k_1}, \dots, x^{k_n} | r_1, \dots, r_n] \in G(\infty, \infty, n)$, we have

$$(2.5.1) \quad \ell(X) = \sum_{\substack{1 \leq i < j \leq n \\ r_i < r_j}} |k_j - k_i| + \sum_{\substack{1 \leq i < j \leq n \\ r_i > r_j}} |k_j - k_i - 1|,$$

where $[c]$ denotes the largest integer not greater than c , and $|c|$ is the absolute value of c for any rational number c .

Proof. Formula (1) is given in [8, Lemma 4.2.2]. Then (2) is a consequence of (1) by applying the map η . \square

2.6. The group $G(\infty, \infty, n)$ can be regarded naturally as a “Weyl group” of some algebraic group $G(A)$. Let \mathbf{k} be the algebraic closure of a prime field \mathbb{F}_p of $p > 0$ elements. Let $A = \mathbf{k}[x, x^{-1}]$, $A^+ = \mathbf{k}[x]$ and $A^- = \mathbf{k}[x^{-1}]$, with x an indeterminate over \mathbf{k} . Let $G = \text{SL}(\mathbf{k}, n)$ be the special linear group of rank n over \mathbf{k} . For any \mathbf{k} -algebra D , denote by $G(D)$ the group $\text{SL}(D, n)$. Let B, B^* be the Borel subgroups of G consisting of all the upper, resp., lower triangular matrices in G . Then the maximal torus $T = B \cap B^*$ of G consists of all the diagonal matrices. Define two homomorphisms $\pi^+ : G(A^+) \rightarrow G$ and $\pi^- : G(A^-) \rightarrow G$ by specializing x and x^{-1} to zero respectively. Then $\tilde{B} = (\pi^+)^{-1}(B)$ and $\tilde{B}^* = (\pi^-)^{-1}(B^*)$ are two opposite Borel subgroups of $G(A)$ with $\tilde{B} \cap \tilde{B}^* = T$. The normalizer $N(A)$ of T consists of all the monomial matrices in $G(A)$. We have $G(\infty, \infty, n) \cong N(A)/T$.

It is an open question to ask if the group $G(m, m, n)$ could also be a “Weyl group” of some algebraic group for any $m, n \in \mathbb{N}$ with $m > 2$.

3. PRESENTATIONS BY GENERATORS AND RELATIONS

In this section, we give two presentations for each of the groups $G(m, m, n)$ and $G(\infty, \infty, n)$: one is standard, and the other is new. We point out some advantages for the new presentations of these groups at the end of the section.

3.1. The group $G(m, m, n)$ is generated by the set $S'(m) = \{t_1, \dots, t_{n-1}, s'\}$ with $t_k = [1, \dots, 1|(k, k+1)]$ and $s' = [e^{-2\pi i/m}, e^{2\pi i/m}, 1, \dots, 1|(1, 2)]$. $G(m, m, n)$ can be presented abstractly by the generator set $S'(m)$ and the following relations (see [2]):

- (a) $t_i^2 = e$ for $i \in [1, n-1]$, where e is the identity element of the group.
- (b) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i \in [1, n-2]$;
- (c) $t_i t_j = t_j t_i$ for $|i - j| > 1$;
- (d) $s'^2 = e$;
- (e) $s' t_2 s' = t_2 s' t_2$;
- (f) $s' t_i = t_i s'$ for $i \in [3, n-1]$.

- (g) $t_1 s' t_2 t_1 s' t_2 = t_2 t_1 s' t_2 t_1 s'$;
 (h) $s' t_1 s' \cdots = t_1 s' t_1 \cdots$ (m factors in both sides).

This presentation can be described by the graph in Fig.1 (see [2]).

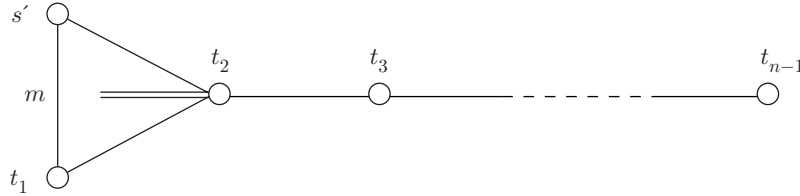


FIGURE 1.

Let $t_0 = t_{n-1} t_{n-2} \cdots t_2 s' t_2 \cdots t_{n-2} t_{n-1}$. Then

$$t_0 = [e^{-2\pi i/m}, 1, \dots, 1, e^{2\pi i/m} | (1, n)].$$

Let $S(m) = \{t_0, t_1, \dots, t_{n-1}\}$. For $1 \leq i < j < n$, let

$$(3.1.1) \quad s_{ij} = t_j t_{j-1} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-1} t_j.$$

Lemma 3.2. *The group $G(m, m, n)$ is generated by the set $S(m)$. Moreover, we have*

- (i) $t_0^2 = e$;
- (ii) $t_0 t_1 t_0 = t_1 t_0 t_1$;
- (iii) $t_0 t_{n-1} t_0 = t_{n-1} t_0 t_{n-1}$;
- (iv) $t_0 t_i = t_i t_0$ for $1 < i < n-1$;
- (v) $t_0 s_{1,n-1} t_0 \cdots = s_{1,n-1} t_0 s_{1,n-1} \cdots$ (m factors in both sides);
- (vi) $t_i t_{i+1} \cdots t_{j-1} t_j t_{j-1} \cdots t_{i+1} t_i = t_j t_{j-1} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-1} t_j$ for $i \neq j$ in $[0, n-1]$, where the subscripts are regarded as the congruence classes of integers modulo n .

Proof. The first assertion follows directly from 3.1. Now we show the others. (i) follows by 3.1 (d) and the fact that t_0 is conjugate to s' . The LHS and RHS of (ii) are equal to

$$t_{n-1} t_{n-2} \cdots t_2 s' t_2 t_1 t_2 s' t_2 \cdots t_{n-2} t_{n-1}$$

and

$$t_{n-1} t_{n-2} \cdots t_3 t_1 t_2 s' t_2 t_1 t_3 \cdots t_{n-2} t_{n-1}$$

respectively. Thus to show (ii), it is enough to show the equation $t_2 s' t_2 t_1 t_2 s' t_2 = t_1 t_2 s' t_2 t_1$. But the latter follows from 3.1 (g). A simple computation shows that the LHS and RHS of (iii) are equal to $s' t_2 t_3 \cdots t_{n-3} t_{n-2} t_{n-3} \cdots t_3 t_2 s'$ and $t_{n-2} t_{n-3} \cdots t_2 s' t_2 \cdots t_{n-3} t_{n-2}$ respectively. They are equal, and hence (iii) follows. (iv) is a consequence of 3.1 (b), (c), (e) and (f). Since

$$t_0 s_{1,n-1} = t_{n-1} t_{n-2} \cdots t_2 s' t_1 t_2 \cdots t_{n-2} t_{n-1},$$

(v) follows from 3.1 (a), (h). Finally, (vi) is a consequence of 3.1 (b), (c), (e), (f) and (ii)-(iv). \square

Note that in the above proof, we only use the relations 3.1 (a)-(h), but not the concrete matrix forms of the elements. Now we give another presentation of the group $G(m, m, n)$.

Proposition 3.3. *The group $G(m, m, n)$ can also be presented abstractly by the generator set $S(m)$ and the following relations.*

- (i) $t_i^2 = e$ for $i \in [0, n-1]$;
- (ii) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i \in [0, n-1]$;
- (iii) $t_i t_j = t_j t_i$ for $i, j \in [0, n-1]$ with $j \not\equiv i \pm 1 \pmod{n}$;
- (iv) $t_0 s_{1,n-1} t_0 \cdots = s_{1,n-1} t_0 s_{1,n-1} \cdots$ (m factors in both sides);

where we stipulate $t_n = t_0$, and for the notation $s_{1,n-1}$ see (3.1.1).

Proof. By Lemma 3.2, it is enough to show that the relations 3.1 (d)-(h) can be deduced from the relations listed in the lemma, where $s' := t_2 t_3 \cdots t_{n-1} t_0 t_{n-1} \cdots t_3 t_2$. Note that the equation in Lemma 3.2 (vi) is a consequence of the relations (ii), (iii). So we can freely use this equation in the subsequent proof. The relation 3.1 (d) follows from (i) since s' is conjugate to t_0 . For 3.1 (e), we have

$$\begin{aligned} s' t_2 &= t_0 t_{n-1} t_{n-2} \cdots t_4 t_3 t_2 t_3 t_4 \cdots t_{n-2} t_{n-1} t_0 t_2 \\ &= t_0 t_{n-1} t_{n-2} \cdots t_4 \cdot t_2 t_3 \cdot t_4 \cdots t_{n-2} t_{n-1} t_0. \end{aligned}$$

Then the relation 3.1 (e) follows from the relations (i), (ii) since $s' t_2$ and $t_2 t_3$ are conjugate. The relation 3.1 (f) follows by (ii), (iii). Now we show the relation 3.1 (g), which is equivalent to the equation

$$\begin{aligned} (3.3.1) \quad & t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 \\ &= t_2 t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 t_2 \end{aligned}$$

The RHS of (3.3.1) is

$$\begin{aligned} & t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_1 t_0 t_{n-1} t_{n-2} \cdots t_3 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 t_2 \\ &= t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_1 \cdot t_2 t_3 \cdots t_{n-1} t_0 t_{n-1} \cdots t_3 t_2 \cdot t_{n-1} t_{n-2} \cdots t_3 t_2 \\ &= t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_1 \cdot t_2 t_3 \cdots t_{n-1} t_0 \cdot t_{n-2} t_{n-3} \cdots t_2 t_{n-1} t_{n-2} \cdots t_4 t_3 \end{aligned}$$

by (i)-(iii). Then to show (3.3.1), it suffices to show that

$$(3.3.2) \quad t_{n-1} t_{n-2} \cdots t_3 t_1 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 = t_1 t_2 t_3 \cdots t_{n-1} t_0 t_{n-2} t_{n-3} \cdots t_2.$$

But the latter can be transformed into

$$t_1 t_{n-1} t_{n-2} \cdots t_3 t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 = t_1 t_2 t_3 \cdots t_{n-1} t_{n-2} t_{n-3} \cdots t_2 t_0$$

or

$$t_{n-1} t_{n-2} \cdots t_3 t_2 t_3 \cdots t_{n-2} t_{n-1} = t_2 t_3 \cdots t_{n-2} t_{n-1} t_{n-2} \cdots t_2.$$

The last equation holds by Lemma 3.2 (vi). This implies 3.1 (g). Finally, the relation 3.1 (h) follows from (iv) by the fact that

$$s' t_1 = t_2 t_3 \cdots t_{n-2} t_{n-1} t_0 t_{n-1} t_{n-2} \cdots t_3 t_2 \cdot t_1$$

is conjugate to $t_0 s_{1,n-1}$. □

The presentation in Proposition 3.3 can be described by the graph in Fig.2.

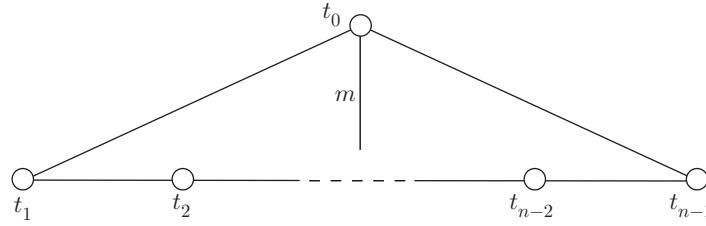


FIGURE 2.

3.4. Next we consider the presentations of the group $G(\infty, \infty, n)$. Let the monomial matrices t_i , $i \in [1, n-1]$, be defined as in 3.1. On the other hand, we redefine two matrices s' , t_0 by $s' = [x^{-1}, x, 1, \dots, 1 | (1, 2)]$ and $t_0 = [x^{-1}, 1, \dots, 1, x | (1, n)]$. Let $S = \{t_0, t_1, \dots, t_{n-1}\}$ and $S' = \{t_1, t_2, \dots, t_{n-1}, s'\}$. Then $G(\infty, \infty, n)$ is generated by the set S or S' . It can be defined abstractly by one of the following presentations.

(1) The generator set S and the relations in Proposition 3.3 (i)-(iii). The corresponding graph is in Fig.3.

(2) The generator set S' and the relations 3.1 (a)-(g). The corresponding graph is in Fig.4.

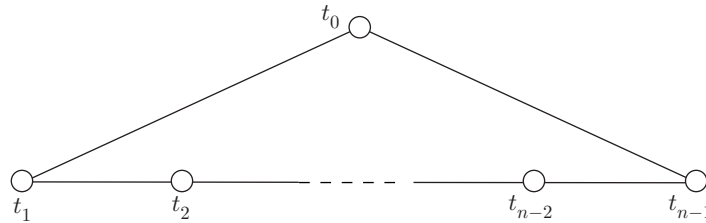


FIGURE 3.

Here presentation (1) is standard by regarding $G(\infty, \infty, n)$ as a Coxeter group (see [5], [8]), while presentation (2) is new by regarding $G(\infty, \infty, n)$ only as a reflection group. These two presentations are compatible with those of $G(m, m, n)$ respectively in the sense that the latter can be regarded as a quotient of the former modulo a certain relation. In particular, the second presentation of $G(\infty, \infty, n)$ has the advantage that a lower rank member in this family is naturally a subgroup of a higher rank member.

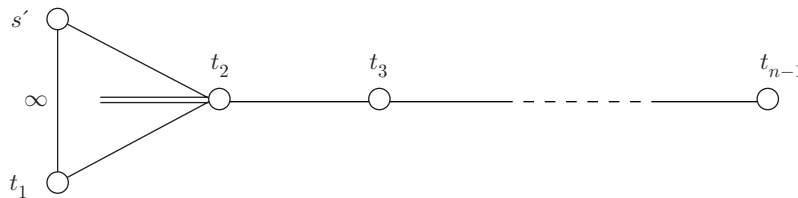


FIGURE 4.

3.5. The presentation of $G(m, m, n)$ in Proposition 3.3 has the advantage that the length function ℓ_m on the elements of $G(m, m, n)$ with respect to the generator set $S(m)$ can be described explicitly. As usual, for $w \in G(m, m, n)$, $\ell_m(w)$ is defined to be the smallest number r such that $w = s_1 s_2 \cdots s_r$ with all s_i in $S(m)$, where we stipulate $\ell_m(e) = 0$ for the identity element e of $G(m, m, n)$. We may assume $m \geq 2$, since a formula for the length function on $G(1, 1, n)$ is already known. Express any $X \in G(m, m, n)$ in the form

$$(3.5.1) \quad X = [\zeta_m^{k_1}, \zeta_m^{k_2}, \dots, \zeta_m^{k_n} | r_1, r_2, \dots, r_n]$$

with $\zeta_m = e^{2\pi i/m}$, where the integers k_j are determined by X up to congruence modulo m , and satisfy $\sum_{j=1}^n k_j \equiv 0 \pmod{m}$. When (k_1, k_2, \dots, k_n) ranges over all the possible integer sequences satisfying (3.5.1) with the additional condition that

$$(3.5.2) \quad \sum_{j=1}^n k_j = 0,$$

$\ell_m(X)$ is the minimal possible value given by the formula (2.5.1). We can even impose one more additional condition on the range of (k_1, k_2, \dots, k_n) in the minimizing process of computing $\ell_m(X)$. That is,

$$(3.5.3) \quad k_l - k_s \leq m + 1,$$

where $l, s \in [1, n]$ satisfy $k_l = \max\{k_j \mid j \in [1, n]\}$ and $k_s = \min\{k_j \mid j \in [1, n]\}$. For, suppose $k_l - k_s \geq m + 2$. Then by replacing k_l, k_s by $k_l - m, k_s + m$ in the formula (2.5.1) respectively, the resulting value will be not greater than the initial one. To see this, it is enough to check the inequality

$$(3.5.4) \quad |k_l - m - k - \epsilon| + |k_s + m - k - \delta| \leq |k_l - k - \epsilon| + |k_s - k - \delta|$$

for any $k_s \leq k \leq k_l$ and $\epsilon, \delta \in \{0, 1, -1\}$. But this is straightforward.

Under the conditions (3.5.1)-(3.5.3), we see that each k_i is in the interval $[-m, m]$ and has at most two possible choices in the calculation of $\ell_m(X)$.

3.6. Let us introduce the associated braid groups of $G(m, m, n)$ and $G(\infty, \infty, n)$. By the same argument as that for Proposition 3.3, we can show that the group presented by

(3.6.1). the generators t_1, \dots, t_{n-1}, s' and the relations 3.1 (b), (c), (e)-(h).
is isomorphic to that presented by

(3.6.2). the generators t_0, t_1, \dots, t_{n-1} and the relations in Proposition 3.3 (ii)-(iv).
Denote both groups by $B(m, m, n)$ (up to isomorphism) and call it the associated braid group of $G(m, m, n)$. Also, the group presented by

(3.6.3). the generators t_0, t_1, \dots, t_{n-1} and the relations in Proposition 3.3 (ii)-(iii)
is isomorphic to that presented by

(3.6.4). the generators t_1, \dots, t_{n-1}, s' and the relations 3.1 (b), (c), (e)-(g).
Denote both groups by $B(\infty, \infty, n)$ (up to isomorphism) and call it the associated braid group of $G(\infty, \infty, n)$.

4. HECKE ALGEBRAS

To two presentations of the group $G(\infty, \infty, n)$ in 3.4, we shall associate Hecke algebras $\mathcal{H}(G(\infty, \infty, n))$ and $\mathcal{H}'(G(\infty, \infty, n))$ respectively. We show that they are actually isomorphic. Accordingly, we shall define two Hecke algebras associated to two presentations of the group $G(m, m, n)$ as certain quotient algebras of $\mathcal{H}(G(\infty, \infty, n))$ and $\mathcal{H}'(G(\infty, \infty, n))$, respectively.

4.1. The Hecke algebra $\mathcal{H}(B(\infty, \infty, n))$ of the braid group $B(\infty, \infty, n)$ with respect to the presentation (3.6.3) is an associative algebra over $\mathcal{B} = \mathbb{Z}[q, q^{-1}]$ (q an indeterminate) with the unity 1, presented by the generator set $\{T_i, T_i^{-1} \mid i \in [0, n-1]\}$ and the relations:

- (1) $T_i^{-1}T_i = T_iT_i^{-1} = 1$ for $i \in [0, n-1]$;
- (2) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $i \in [0, n-1]$;
- (3) $T_iT_j = T_jT_i$ for $i \not\equiv j \pm 1 \pmod{n}$,

where we stipulate $T_n = T_0$. Then the Hecke algebra $\mathcal{H}(G(\infty, \infty, n))$ of the group $G(\infty, \infty, n)$ with respect to the presentation 3.4 (1) is an associative algebra over \mathcal{B} with the unity 1, presented by the generator set $\{T_i \mid i \in [0, n-1]\}$ and the relations (2)-(4), where

- (4) $(T_i - q)(T_i + 1) = 0$ for $i \in [0, n-1]$.

Note that all the T_i 's are invertible in $\mathcal{H}(G(\infty, \infty, n))$.

4.2. We can define another Hecke algebra $\mathcal{H}'(B(\infty, \infty, n))$ of the group $B(\infty, \infty, n)$ with respect to the presentation (3.6.4). This is an associative algebra over \mathcal{B} with the unity 1 presented by the generator set $\{T'_i, T'^{-1}_i, S', S'^{-1} \mid i \in [1, n-1]\}$ and the relations:

- (i) $T'^{-1}_iT'_i = T'_iT'^{-1}_i = 1$ for $i \in [1, n-1]$;
- (ii) $S'S'^{-1} = 1$;
- (iii) $T'_iT'_{i+1}T'_i = T'_{i+1}T'_iT'_{i+1}$ for $i \in [1, n-2]$;
- (iv) $T'_iT'_j = T'_jT'_i$ for $|j - i| > 1$;
- (v) $S'T'_2S' = T'_2S'T'_2$;
- (vi) $S'T'_j = T'_jS'$ for $j \in [3, n-1]$;
- (vii) $T'_1S'T'_2T'_1S'T'_2 = T'_2T'_1S'T'_2T'_1S'$.

Also, another Hecke algebra $\mathcal{H}'(G(\infty, \infty, n))$ of the group $G(\infty, \infty, n)$ with respect to the presentation 3.4 (2) is an associative algebra over \mathcal{B} with the unity 1 presented by the generator set $\{T'_i, S' \mid i \in [1, n-1]\}$ and the relations (iii)-(ix), where

- (viii) $(T'_i - q)(T'_i + 1) = 0$ for $i \in [1, n-1]$;
- (ix) $(S' - q)(S' + 1) = 0$.

Clearly, S' and T'_i , $i \in [1, n-1]$, are invertible in $\mathcal{H}'(G(\infty, \infty, n))$.

We have the following relations among these Hecke algebras.

Proposition 4.3. (1) $\mathcal{H}'(B(\infty, \infty, n)) \cong \mathcal{H}(B(\infty, \infty, n))$;
 (2) $\mathcal{H}'(G(\infty, \infty, n)) \cong \mathcal{H}(G(\infty, \infty, n))$.

Proof. We first show (1). Set $\phi(T'^{\pm 1}_i) = T_i^{\pm 1}$ for $i \in [1, n-1]$ and $\phi(S'^{\pm 1}) = XT_0^{\pm 1}X^{-1}$, where $X = T_2T_3 \cdots T_{n-1}$. We want to show that ϕ can be extended to an algebra homomorphism from $\mathcal{H}'(B(\infty, \infty, n))$ to $\mathcal{H}(B(\infty, \infty, n))$. It is enough to show the following relations.

- (a) $XT_0X^{-1} \cdot T_2 \cdot XT_0X^{-1} = T_2 \cdot XT_0X^{-1} \cdot T_2$;
- (b) $T_1 \cdot XT_0X^{-1} \cdot T_2T_1 \cdot XT_0X^{-1} \cdot T_2 = T_2T_1 \cdot XT_0X^{-1} \cdot T_2T_1 \cdot XT_0X^{-1}$;

(c) $XT_0X^{-1} \cdot T_j = T_j \cdot XT_0X^{-1}$ for $j \in [3, n-1]$.

Applying the equation

$$(4.3.1) \quad T_j^{-1}T_{j-1}^{-1} \cdots T_{i+1}^{-1}T_iT_{i+1} \cdots T_j = T_iT_{i+1} \cdots T_{j-1}T_jT_{j-1}^{-1} \cdots T_i^{-1}$$

for $1 \leq i < j \leq n-1$, we see that the equation (a) is equivalent to

$$\begin{aligned} & T_2T_3 \cdots T_{n-1}T_0 \cdot T_2T_3 \cdots T_{n-2}T_{n-1}T_{n-2}^{-1} \cdots T_3^{-1}T_2^{-1} \cdot T_0 \\ &= T_2^2T_3 \cdots T_{n-1}T_0T_{n-1}^{-1} \cdots T_3^{-1} \cdot T_2T_3 \cdots T_{n-1}. \end{aligned}$$

This is further equivalent to

$$T_3 \cdots T_{n-1}T_0 \cdot T_2T_3 \cdots T_{n-2}T_{n-1}T_0 = T_2T_3 \cdots T_{n-1}T_0T_2T_3 \cdots T_{n-2}T_{n-1}.$$

But the last equation follows by applying the relations $T_k \cdot XT_0 = XT_0 \cdot T_{k-1}$ repeatedly for $k \in \{0, 3, 4, \dots, n-1\}$. Next by applying the equation $T_2T_1X = T_1XT_1$, we see that the equation (b) is equivalent to

$$T_0X^{-1}T_1XT_1T_0X^{-1}T_2X = T_1T_0X^{-1}T_1XT_1T_0,$$

which is further equivalent to

$$\begin{aligned} & T_0 \cdot T_1T_2 \cdots T_{n-2}T_{n-1}T_{n-2}^{-1} \cdots T_1^{-1} \cdot T_1T_0 \cdot T_2T_3 \cdots T_{n-2}T_{n-1}T_{n-2}^{-1} \cdots T_3^{-1}T_2^{-1} \\ &= T_1T_0 \cdot T_1T_2 \cdots T_{n-2}T_{n-1}T_{n-2}^{-1} \cdots T_2^{-1}T_1^{-1} \cdot T_1T_0 \end{aligned}$$

by repeatedly applying (4.3.1). But the last equation can be simplified into

$$T_{n-1}T_0T_{n-1} = T_0T_{n-1}T_0$$

after a certain cancelation, which is obviously true. This implies (b).

The equation (c) follows easily by the facts $XT_{j-1} = T_jX$ and $T_{j-1}T_0 = T_0T_{j-1}$ for $j \in [3, n-1]$. So ϕ can be extended to an algebra homomorphism from $\mathcal{H}'(B(\infty, \infty, n))$ to $\mathcal{H}(B(\infty, \infty, n))$; the latter is still denoted by ϕ . Now we shall show that ϕ is an isomorphism. Define $\phi'(T_i^{\pm 1}) = T_i'^{\pm 1}$ for $i \in [1, n-1]$ and $\phi'(T_0^{\pm 1}) = Y^{-1}S'^{\pm 1}Y$, where $Y = T_2'T_3' \cdots T_{n-1}'$. We want to show that ϕ' can be extended to the inverse map of the homomorphism ϕ . It is enough to show the following relations.

(d) $Y^{-1}S'Y \cdot T_j' \cdot Y^{-1}S'Y = T_j' \cdot Y^{-1}S'Y \cdot T_j'$ for $j = 1, n-1$.

(e) $Y^{-1}S'Y \cdot T_k = T_k \cdot Y^{-1}S'Y$ for $k \in [2, n-2]$.

When $j = 1$, the equation (d) is equivalent to

$$T_2'^{-1}S'T_2'T_1'T_2'^{-1}S'T_2' = T_1'T_2'^{-1}S'T_2'T_1',$$

which follows from 4.2 (vii). When $j = n-1$, (d) is equivalent to

$$\begin{aligned} & T_{n-1}'^{-1} \cdots T_2'^{-1}S'T_2' \cdots T_{n-2}'T_{n-1}'T_{n-2}'^{-1} \cdots T_2'^{-1}S'T_2' \cdots T_{n-1}' \\ &= T_{n-2}'^{-1} \cdots T_2'^{-1}S'T_2' \cdots T_{n-2}'T_{n-1}'. \end{aligned}$$

The latter can be shown by repeatedly applying the relations

$$T_k'^{-1} \cdots T_2'^{-1}S'T_2' \cdots T_k' = S'T_2' \cdots T_{k-1}'T_k'T_{k-1}'^{-1} \cdots T_2'^{-1}S'^{-1}$$

for $k \in [2, n-1]$. Finally, the equation (e) follows by the facts that $YT_k = T_{k+1}Y$ and $T_{k+1}S' = S'T_{k+1}$ for $k \in [2, n-2]$.

So (1) is shown. For (2), let $\phi(T'_i) = T_i$, $\phi(T_i) = T'_i$ for $i \in [1, n-1]$, and $\phi(S') = XT_0X^{-1}$, $\phi'(T_0) = Y^{-1}S'Y$ with X, Y as before. Then we can show (2) by the same argument as that for (1), and by checking the relations $(\phi(S') - q)(\phi(S') + 1) = 0$,

$(\phi(T'_i) - q)(\phi(T'_i) + 1) = 0$ and $(\phi'(T_j) - q)(\phi'(T_j) + 1) = 0$ for $i \in [1, n-1]$ and $j \in [0, n-1]$ \square

By Proposition 4.3, we can identify $\mathcal{H}(B(\infty, \infty, n))$ with $\mathcal{H}'(B(\infty, \infty, n))$ (resp. $\mathcal{H}(G(\infty, \infty, n))$ with $\mathcal{H}'(G(\infty, \infty, n))$) and call it the Hecke algebra of the group $B(\infty, \infty, n)$ (resp. $G(\infty, \infty, n)$). Clearly, the Hecke algebra of $G(\infty, \infty, n)$ is a quotient of that of $B(\infty, \infty, n)$.

4.4. The Hecke algebra $\mathcal{H}(B(m, m, n))$ (resp. $\mathcal{H}(G(m, m, n))$) of $B(m, m, n)$ (resp. $G(m, m, n)$) also has two equivalent presentations: one is as the quotient of $\mathcal{H}(B(\infty, \infty, n))$ (resp. $\mathcal{H}(G(\infty, \infty, n))$) modulo the relation

$$\underbrace{T_0 R T_0 \cdots \cdots}_{m \text{ factors}} = \underbrace{R T_0 R \cdots \cdots}_{m \text{ factors}},$$

where $R = T_1 T_2 \cdots T_{n-2} T_{n-1} T_{n-2}^{-1} \cdots T_2^{-1} T_1^{-1}$. The other is as the quotient of $\mathcal{H}'(B(m, m, n))$ (resp. $\mathcal{H}'(G(m, m, n))$) modulo the relation

$$\underbrace{T'_1 S' T'_1 \cdots \cdots}_{m \text{ factors}} = \underbrace{S' T'_1 S' \cdots \cdots}_{m \text{ factors}}.$$

5. THE ROOT SYSTEMS OF $G(\infty, \infty, n)$ AND $G(m, m, n)$

We shall give a new description for the root system $\overline{\Phi}$ of the affine Weyl group $G(\infty, \infty, n)$. The action of $G(\infty, \infty, n)$ on $\overline{\Phi}$ is natural and compatible with that of $G(m, m, n)$ on its root system $\Phi(m)$ via the pre-root system Φ and the specialization $x = e^{2\pi i/m}$.

5.1. Let e_1, \dots, e_n be the canonical basis of the vector space \mathbb{C}^n . Let $\zeta_m = e^{2\pi i/m}$ and $\delta_{i,j;h,k} = \frac{1}{\sqrt{2}}(\zeta_m^h e_i - \zeta_m^k e_j)$. Then

$$\Phi(m) = \{\delta_{i,j;h,k} \mid i \neq j \text{ in } [1, n]; h, k \in [0, m-1]\}$$

forms a root system of $G(m, m, n)$ on which $G(m, m, n)$ acts naturally (see [3]).

5.2. We shall give a new description for the root system of $G(\infty, \infty, n)$. Let $\mathcal{A} = \mathbb{C}[x, x^{-1}]$ be the ring of the Laurent polynomials in an indeterminate x with complex coefficients. Define an involution $\bar{\cdot}$ in \mathcal{A} by $\overline{\sum_i a_i x^i} = \sum_i \overline{a_i} x^{-i}$, where $\overline{a_i}$ is the complex conjugate of $a_i \in \mathbb{C}$.

Let E be a free \mathcal{A} -module with the basis e_1, \dots, e_n . Then E has a complex space structure of infinite dimension which contains \mathbb{C}^n as a subspace. Define a pairing $(\mid) : E \times E \rightarrow \mathcal{A}$ by setting $(f \mid g) = \sum_k f_k \overline{g_k}$ for $f = \sum_i f_i e_i$, $g = \sum_j g_j e_j$ in E . It is \mathcal{A} -linear in the first factor f and \mathcal{A} -semilinear in the second factor g . Its restriction to \mathbb{C}^n is a unitary inner product.

Let Φ be the set of all the elements $\gamma_{i,j;h,k} = \frac{1}{\sqrt{2}}(x^h e_i - x^k e_j)$ in E with $i \neq j$ in $[1, n]$ and $h, k \in \mathbb{Z}$. Clearly, we have $(\alpha \mid \alpha) = 1$ for $\alpha \in \Phi$. Define $s'_\alpha : E \rightarrow E$ by $s'_\alpha(v) = v - 2(v \mid \alpha)\alpha$, $v \in E$, called a reflection in E with respect to α . The action of s'_α on E induces a permutation on Φ .

Write $v \sim_x v'$ in E if $v' = x^k v$ for some $k \in \mathbb{Z}$. This defines an equivalence relation in E . The corresponding equivalence classes of E are called x -classes. We have $\gamma_{i,j;h,k} \sim_x \gamma_{i,j;h+1,k+1}$ for any $h, k \in \mathbb{Z}$ and $i \neq j$ in $[1, n]$. Let \overline{E} be the set of x -classes in E . We see that Φ is a union of x -classes in E . Let $\overline{\Phi}$ be the set of x -classes of E in Φ . The action of s'_α on E preserves x -classes, and hence induces a permutation on the set $\overline{\Phi}$.

Denote by $\bar{\alpha}$ the x -class of E containing α . Denote $\gamma_{i,j;0,k}$ by $\gamma_{i,j;k}$. Then $\Phi_0 = \{\gamma_{i,j;k} \mid i \neq j \text{ in } [1, n]; k \in \mathbb{Z}\}$ forms a representative set of $\bar{\Phi}$ in Φ .

Denote $s'_i = s'_{\gamma_{i,i+1;0}}$, $i \in [1, n-1]$, and $s'_0 = s'_{\gamma_{n,1;1}}$. Then for $i \in [0, n-1]$, the matrix of s'_i with respect to the basis e_1, \dots, e_n is just t_i in 3.4. By identifying s'_i with t_i , $i \in [0, n-1]$, the group $G(\infty, \infty, n)$ naturally acts on E and hence on the set $\bar{\Phi}$.

We see that $(\overline{\gamma_{i,j;p}} + \overline{\gamma_{k,l;q}}) \cap \Phi \neq \emptyset$ if and only if either $j = k \neq i \neq l \neq j$ or $i = l \neq j \neq k \neq i$ holds. When the equivalent conditions hold, $(\overline{\gamma_{i,j;p}} + \overline{\gamma_{k,l;q}}) \cap \Phi$ is a single x -class of E , written $\bar{\gamma}$. More precisely, when $j = k \neq i \neq l \neq j$ (resp. $i = l \neq j \neq k \neq i$), we have $\bar{\gamma} = \overline{\gamma_{i,l;p+q}}$ (resp. $\bar{\gamma} = \overline{\gamma_{k,j;p+q}}$). We regard $\bar{\gamma}$ as the sum of $\overline{\gamma_{i,j;p}}$ and $\overline{\gamma_{k,l;q}}$.

5.3. The following description of the affine root system of \tilde{A}_{n-1} is due to Lusztig (see [7]). Let $R = \{\alpha_{ij} \mid i \neq j \text{ in } [1, n]\}$ be the root system of S_n satisfying $-\alpha_{ij} = \alpha_{ji}$ and $\alpha_{it} + \alpha_{tj} = \alpha_{ij}$. Then $\Pi = \{\alpha_{i,i+1} \mid i \in [1, n-1]\}$ forms a simple root system of R . The sets $R^+ = \{\alpha_{ij} \in R \mid i < j\}$ and $R^- = \{\alpha_{ij} \in R \mid i > j\}$ are the corresponding positive and negative root systems of R respectively. Let $X = \mathbb{Z}R$ be the root lattice.

The root system of the affine Weyl group $\tilde{A}_{n-1} = S_n \ltimes X$ is $\tilde{R} = \{(\alpha, k) \mid \alpha \in R; k \in \mathbb{Z}\}$. For $p, q \in \mathbb{Z}$ and distinct $i, j, k \in [1, n]$, we have $(\alpha_{ik}, p + q) = (\alpha_{ij}, p) + (\alpha_{jk}, q)$ and $-(\alpha_{ij}, p) = (\alpha_{ji}, -p)$. The set

$$\tilde{\Pi} = \{(\alpha_{i,i+1}, 0) \mid i \in [1, n-1]\} \cup \{(\alpha_{n,1}, 1)\}$$

is a simple root system of \tilde{R} . The corresponding positive and negative root systems of \tilde{R} are $\tilde{R}^+ = \{(\alpha, k) \mid \alpha \in R, k > 0\} \cup \{(\alpha, 0) \mid \alpha \in R^+\}$ and $\tilde{R}^- = \{(\alpha, k) \mid \alpha \in R, k < 0\} \cup \{(\alpha, 0) \mid \alpha \in R^-\}$, respectively. Denote $\lambda \in X$ by T_λ as an element of \tilde{A}_{n-1} . An element $wT_\lambda \in \tilde{A}_{n-1}$ acts on \tilde{R} by sending (α, k) to $(w(\alpha), k - \langle \lambda, \alpha^\vee \rangle)$, where $\langle \cdot, \cdot \rangle$ is the inner product of the euclidean space spanned by R . Denote $\alpha_{i,j;k} = (\alpha_{ij}, k)$ for $i \neq j$ in $[1, n]$ and $k \in \mathbb{Z}$. The reflection with respect to $\alpha_{i,j;h}$ is $s_{\alpha_{i,j;h}} = s_{\alpha_{ij}} T_{h\alpha_{ij}}$. Let $s_i = s_{\alpha_{i,i+1;0}}$, $i \in [1, n-1]$, and $s_0 = s_{\alpha_{n,1;1}}$. Then $S = \{s_i \mid i \in [0, n-1]\}$ is a distinguished generator set of \tilde{A}_{n-1} which satisfies the relations in Proposition 3.3 (i)-(iii) with s_i in the place of t_i .

5.4. Define $\psi : \tilde{R} \longrightarrow \Phi_0$ by $\psi(\alpha_{i,j;k}) = \gamma_{i,j;k}$ for $\alpha_{i,j;k} \in \tilde{R}$. This is a bijection, which induces a natural bijection: $\tilde{R} \longrightarrow \bar{\Phi}$ (still denoted by ψ). Clearly, ψ respects the additions whenever it is applicable. Moreover, under the identification of $G(\infty, \infty, n)$ with \tilde{A}_{n-1} , we can denote both $s_i \in \tilde{A}_{n-1}$ and $s'_i \in G(\infty, \infty, n)$ by t_i for $i \in [0, n-1]$. Then we have

$$\psi(t_i(\alpha)) = t_i(\psi(\alpha)) \quad \text{for } \alpha \in \tilde{R}, i \in [0, n-1].$$

So the map ψ also respects the group actions. We call $\bar{\Phi}$ the root system of $G(\infty, \infty, n)$.

5.5. Under the specialization $x = e^{2\pi i/m}$, the \mathcal{A} -module E becomes the complex vector space \mathbb{C}^n , the group $G(\infty, \infty, n)$ becomes $G(m, m, n)$, and the set Φ becomes $\Phi(m)$. Also, the set $\{\gamma_{i,j;k} \mid j < i \text{ in } [1, n]; k \in \mathbb{Z}\}$ becomes the set Ω_2 defined in [1] whose role in $\Phi(m)$ is analogous to a positive root system of a Coxeter group. The action of $G(\infty, \infty, n)$ on Φ is compatible with that of $G(m, m, n)$ on $\Phi(m)$ under this specialization.

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